

Theoretical basis

1D propagation model of S waves into a horizontal layered subsoil.

Vertical propagation of shear waves, with frequency ω , causes horizontal displacements $u(z,t)$, which must satisfy the equation:

$$(1) \rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} + \eta \frac{\partial^3 u}{\partial z^2 \partial t}$$

Harmonic displacements u with frequency ω , can be expressed in the following form too:

$$(2) u(z,t) = U(z)e^{i\omega t}$$

Substituting (2) into (1):

$$(3) (G + i\omega\eta) \frac{\partial^2 U}{\partial z^2} = \rho\omega^2 U$$

where ρ is the mass density and G is the shear modulus of the soil layer. Equation (3) has the following general solution:

$$(4) U(z) = Ee^{ikz} + Fe^{-ikz}$$

where:

$$k = \sqrt{\frac{\rho\omega^2}{G^*}}$$

G^* is the complex shear modulus:

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$$G^* = G(1 + 2i\beta)$$

and β is the critical damping factor:

$$\beta = \frac{\omega\eta}{2G}$$

In equation (4) E is the incident wave traveling upwards and F is the reflected wave traveling downwards.

Combine (2) and (4):

$$(5) u(z, t) = (Ee^{ikz} + Fe^{-ikz})e^{i\omega t}$$

In a multilayer subsoil, at the top of the layer n , with thickness h , the displacements are:

$$(6) u_n(z = 0) = (E_n + F_n)e^{i\omega t}$$

at the bottom:

$$(7) u_n(z = h) = (E_n e^{ik_n h_n} + F_n e^{-ik_n h_n})e^{i\omega t}$$

Shear stresses acting along the horizontal plane at the top and bottom of the layer are respectively:

$$(8) \tau_n(z = 0) = ik_n G_n^* (E_n + F_n)e^{i\omega t}$$

$$(9) \tau_n(z = h) = ik_n G_n^* (E_n e^{ik_n h_n} + F_n e^{-ik_n h_n})e^{i\omega t}$$

Shear strains ($\gamma(z, t) = \frac{\partial u}{\partial t}$) are instead:

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$$\begin{aligned}\gamma_n(z=0) &= ik_n (E_n + F_n) e^{i\omega t} \\ \gamma_n(z=h) &= ik_n (E_n e^{ik_n h_n} + F_n e^{-ik_n h_n}) e^{i\omega t}\end{aligned}$$

In a multi-layer subsoil the parameters ρ , G e β change generally with the depth. Thus E and F have different values. Stresses and displacements must be continuous at all interfaces. Amplitudes E and F of the incident and reflected wave of the layer n can be expressed in the following way:

$$(10) E_{n+1} = \frac{1}{2} E_n (1 + \alpha_n) e^{ik_n h_n} + \frac{1}{2} F_n (1 - \alpha_n) e^{-ik_n h_n}$$

$$(11) F_{n+1} = \frac{1}{2} E_n (1 - \alpha_n) e^{ik_n h_n} + \frac{1}{2} F_n (1 + \alpha_n) e^{-ik_n h_n}$$

where:

$$\alpha_n = \sqrt{\frac{\rho_n G_n^*}{\rho_{n+1} G_{n+1}^*}}$$

is the complex impedance ratio.

At the ground surface the shear stress has to be zero. From equation (8):

$$E_1 = F_1.$$

For the case $E_1 = F_1 = 1$, the amplitudes E and F of the layer n can be determined by substituting this condition in the formulas (10) and (11), starting from the ground surface to the bedrock.

Transfer function between the displacements at level n and $n+1$ is defined by:

$$(12) A_{n+1,n}(\omega) = \frac{u_n}{u_{n+1}} = \frac{E_n + F_n}{E_{n+1} + F_{n+1}}$$

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At the bedrock interface $E'=F'$ (shear stress=0). As a result transfer function of the shear wave at the ground surface in respect to the bedrock is given by:

$$(13) A_{bedrock,1}(\omega) = \frac{1}{E_{bedrock}}$$

Evaluation of the seismic motion on the ground surface.

The calculation procedure seen in the previous paragraph is referred to the case of a single wave with angular frequency ω .

An accelerogram is composed by a set of acceleration values a inside a interval of recording time T . The a values are sampled with a constant time step Δt , sampling interval, usually 0.01 or 0.02 seconds. Thus the duration of the earthquake is given by:

$$(14) T(s) = n\Delta t$$

where n is the total number of the sampling intervals.

A seismic pulse, recorded as an accelerogram, can be considered as a finite sum of waves having different frequency ω and thus can be expressed in the form of a discrete Fourier transform.

$$(15) a(t) = \sum_{j=0}^{j=n/2} (a_j e^{i\omega_j t} + b_j e^{-i\omega_j t})$$

where $n/2$ are the angular frequencies given by:

$$\omega_j = \frac{2\pi}{n\Delta t} j$$

The variables a and b , as a function of ω , known as complex coefficients, are gotten through the following formulas:

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$$(16) a_j = \frac{1}{n} \sum_{k=1}^{n-1} a(t) e^{-i\omega_j t}$$

$$(17) b_j = \frac{1}{n} \sum_{k=1}^{n-1} a(t) e^{i\omega_j t}$$

Practically, the values of a and b are calculated, using the fast Fourier transform (FFT), and graphically displayed as Fourier's spectra. Particularly, the amplitude spectrum is given by the following relation:

$$(18) |F(\omega)|_j = \sqrt{a_j^2 + b_j^2}$$

The calculation of the amplification factor, for each values of ω , permits to generate the transfer function of the seismic motion from the bedrock to the ground surface. If the formula (15) refers to the recorded accelerogram on the bedrock, the one at the ground surface is given by the following expression:

$$(19) a_{\text{superficie}}(t) = \sum_{j=0}^{j=n/2} A_{\text{bedrock},1}(\omega_j) (a_j e^{i\omega_j t} + b_j e^{-i\omega_j t})$$

The $a_{\text{superficie}}$ values, throughout the duration T of the earthquake, are computed using the inverse fast Fourier transform (IFFT) applied to the (19).

Evaluation of the stratigraphic seismic amplification through an equivalent linear model

The calculation procedure illustrated in the previous paragraph is applicable in case of nonlinear behavior of the subsoil too. A nonlinear behavior usually arises in case of earthquake with peak acceleration higher than 0.1 g. Both the shear modulus G and the damping factor β of the subsoil change as a function of the strain level due to the seismic stress. This phenomenon can be taken in account using a linear equivalent model. The procedure is the following:

1. A first seismic response is calculated inserting G_0 and β_0 , the shear modulus and the damping factor for low strain, of every soil layer.
2. The shear strain, due to the seism, is evaluated; then G and β are updated on the basis of the $G(\gamma)$ and $\beta(\gamma)$ curves (see the next paragraph).
3. The steps 1 and 2 have to be repeated till when the difference between the shear strains calculated in two consecutive iterations is below a predetermined limit.

Evaluation of the $G(\gamma)$ and $\beta(\gamma)$ curves.

The $G(\gamma)$ and $\beta(\gamma)$ curves are usually gotten through laboratory tests (resonant column test). As an alternative it's possible to use the empirical curves, as a function of the lithology, suggested by several authors.

Clay (Seed & Idriss)

γ (%)	G/G_0	β (%)
0.0001	1	0.24
0.0003	1	0.42
0.001	1	0.8
0.003	0.981	1.4
0.01	0.941	2.8
0.03	0.847	5.1
0.1	0.656	9.8
0.3	0.438	15.5
1	0.238	21
3	0.144	25
10	0.11	28

Sand (Seed & Idriss)

γ (%)	G/G_{max}	β (%)
0.0001	1	0.24
0.0003	1	0.42
0.001	0.99	0.8
0.003	0.96	1.4
0.01	0.85	2.8
0.03	0.64	5.1
0.1	0.37	9.8
0.3	0.18	15.5
1	0.08	21
3	0.05	25
10	0.035	28

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Rock (Seed e Idriss)

γ (%)	G/G _{max}	β (%)
0.0001	1	0.4
0.0003	1	0.8
0.001	0.9875	1.5
0.003	0.9525	3
0.01	0.9	4.6
0.03	0.81	
0.1	0.725	
1	0.55	

Gravel and pebbles-Gravel and sand (Rollins)

γ (%)	G/G _{max}	β (%)
0.0001	1	1
0.0003	1	1
0.001	0.96	1.5
0.003	0.88	2.1
0.01	0.75	4
0.03	0.55	7
0.1	0.33	11
0.3	0.19	14
1	0.05	17

Gravelly clay-Silty clay(Rollins)

γ (%)	G/G _{max}	β (%)
0.0001	1	2
0.0003	1	2
0.001	0.97	2.1
0.003	0.95	2.2
0.01	0.88	3
0.03	0.70	4.5
0.1	0.37	10
0.3	0.15	17
1	0.05	25

Calculation of the elastic response spectra.

The seismic response of a building is often simplified using a single degree-of-freedom system (SDOF). Such an oscillator, staying on the ground in an initial condition of stillness [$y(0)=0$], and then forced to move by an earthquake, will be subject to a displacement y , as a function of the time, given by :

$$y(t) = -\frac{1}{\omega_d} \int_0^t a(\tau) e^{-\zeta\omega_n(t-\tau)} \text{sen}[\omega_d(t-\tau)] d\tau$$

where the integral at the second member is known as Duhamel's integral and where:

- ζ = damping factor of the oscillator, normally lower than 0.1 (10 %) and usually equal to 0.05 (5%);
- ω_n = natural oscillation period, given by $2\pi/T$;
- ω_d = damped oscillation period, given by $\omega_n \sqrt{1-\zeta^2}$.

The elastic response spectra of the acceleration is given through the following steps:

1. The damping factor of the oscillator (normally 5%) and the interval of periods (usually 0-4 seconds) are imposed.
2. ω_n e ω_d are calculated for each value of T .
3. For each value of t , t varying from 0 (start of the recorded accelerogram) till t_{sisma} (end of the recording), the Duhamel's integral is solved.
4. The numerical solution gives n displacements y for every period T ; drawing the maximum calculated value of y (y_{max}) as a function of T , the elastic response spectrum of the acceleration is obtained.
5. Starting from the spectrum of the acceleration, it's possible to estimate the spectra of velocity and displacement (PSV and PSD), simply multiplying the graphical ordinates by ω_n e ω_n^2 .

It's important to notice that the link between accelerogram and spectra is not biunivocal: different accelerograms could give similar spectra.

Elastic response spectra allow to estimate, in a fast and simply way, the seismic stress acting on the building, owning a natural period of oscillation

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$T > 0$, during an earthquake (for $T = 0$ the spectrum gives the acceleration of the ground).

Elastic response spectra, generated by an accelerogram, have an irregular shape. To simplify the using, the spectrum is usually modified through common techniques of smoothing (for example simple or exponential moving average).

It's possible to generate a regular spectrum, using as a reference the coordinates of the maximum ordinate of the calculated spectrum (T_{max} , PSA_{max}), employing the following parameters:

$S_s = FAD =$ maximum amplification factor of the accelerations;

$S_t = 1$;

$T_c = 2\pi(F_v/F_a)$;

$T_b = T_c/3$;

$T_d = 4(a_{gbedrock} \times S_s \times S_t) + 1,6$

$F_0 = PSA_{max} / (a_{gbedrock} \times S_s \times S_t \times n)$;

where n is a variable associated to the damping factor of the building:

$$\eta = \left[\frac{10}{5 + \xi} \right]^{0.5}$$

F_a and F_v (respectively the spectral amplification of acceleration and velocity) are evaluated through a comparison between the response spectra calculated at the ground surface and at the bedrock.

The steps to calculate F_a are the following:

1. the period T , corresponding to the maximum value of acceleration in the spectrum of the bedrock, is identified (T_{max});
2. the mean value of the ordinate around T_{max} is estimated by the following relation:

$$A_{medio} = \frac{1}{T_{max}} \int_{0,5T_{max}}^{1,5T_{max}} A(T) dT$$

3. the steps 1 and 2 are repeated for the surface spectrum of the acceleration;
4. F_a is calculated as the ratio between A_{medio} at the ground surface and at the bedrock:

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$$Fa = \frac{A_{mediosuperficie}}{A_{mediobedrock}}$$

The steps to calculate Fv are the following:

5. the period T, corresponding to the maximum value of velocity in the spectrum of the bedrock, is identified (T_{max});
6. the mean value of the ordinate around T_{max} is estimated by the following relation:

$$V_{medio} = \frac{1}{T_{max}} \int_{0,8T_{max}}^{1,2T_{max}} V(T) dT$$

7. the steps 1 and 2 are repeated for the surface spectrum of the velocity;
8. Fv is calculated as the ratio between V_{medio} at the ground surface and at the bedrock:

$$Fv = \frac{V_{medio\ superficie}}{V_{mediobedrock}}$$